SPACE-LIKE AND TIME-LIKE HYPERSPHERES IN REAL PSEUDO-RIEMANNIAN 4-SPACES WITH ALMOST CONTACT B-METRIC STRUCTURES

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Abstract. There are considered 4-dimensional pseudo-Riemannian spaces with inner products of signature (3,1) and (2,2). The objects of investigation are space-like and time-like hyperspheres in the respective cases. These hypersurfaces are equipped with almost contact B-metric structures. The constructed manifolds are characterized geometrically.

Introduction

The geometry of 4-dimensional Riemannian spaces is well developed. When the metric is generalized to pseudo-Riemannian there are two significant cases: the Lorentz-Minkowski space $\mathbb{R}^{3,1}$ and the neutral pseudo-Euclidean 4-space $\mathbb{R}^{2,2}$. These spaces are object of special interest because of their importance in physics. The space $\mathbb{R}^{3,1}$ has applications in the general relativity and the space $\mathbb{R}^{2,2}$ is connected to the string theory.

Hyperspheres in an even-dimensional space are known as a fundamental example of almost contact metric manifolds (cf. [1]). We are interested in almost contact B-metric structures, introduced in [3]. In the present work we consider space-like and time-like hyperspheres in $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$, known also as 3-dimensional de Sitter and anti-de Sitter space-times, respectively (cf. [2]). After that we construct almost contact B-metric manifolds on these hypersurfaces. Then we study some their geometrical properties.

The paper[1] is organized as follows. In Sect. 1 we recall some preliminary facts about the considered manifolds. In Sect. 2 we are interested in space-like spheres in $\mathbb{R}^{3,1}$. Sect. 3 is devoted to time-like spheres in $\mathbb{R}^{2,2}$.

1. Preliminaries

Let us denote an *almost contact B-metric manifold* by $(M, \varphi, \xi, \eta, g)$, i.e. M is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure (φ, ξ, η) consisting of an endomorphism φ of the tangent bundle, a Reeb vector field ξ, its dual contact 1-form η as well as M is equipped with a pseudo-Riemannian metric g of signature $(n+1, n)$, such that the following algebraic relations are

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satisfied [3]:

$$\begin{gathered}\varphi\xi = 0, \qquad \varphi^2 = -\mathrm{Id} + \eta\otimes\xi, \qquad \eta\circ\varphi = 0, \qquad \eta(\xi) = 1,\\ g(\varphi x, \varphi y) = -g(x,y) + \eta(x)\eta(y),\end{gathered}$$

where Id is the identity. In the latter equality and further, x, y, z, w will stand for arbitrary elements of $\mathfrak{X}(M)$, the Lie algebra of tangent vector fields, or vectors in the tangent space T_pM of M at an arbitrary point p in M.

A classification of almost contact B-metric manifolds, consisting of eleven basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\dots$, $\mathcal{F}_{11}$, is given in [3]. This classification is made with respect to the tensor F of type (0,3) defined by

$$F(x,y,z) = g\big((\nabla_x\varphi)\,y, z\big),$$

where ∇ is the Levi-Civita connection of g. The following properties are valid in general:

$$\begin{aligned}&F(x,y,z) = F(x,z,y) = F(x,\varphi y,\varphi z) + \eta(y)F(x,\xi,z) + \eta(z)F(x,y,\xi),\\ &F(x,\varphi y,\xi) = (\nabla_x\eta)y = g(\nabla_x\xi, y).\end{aligned} \tag{1.1}$$

The intersection of the basic classes is the special class $\mathcal{F}_0$, determined by the condition $F(x,y,z) = 0$, and it is known as the class of the *cosymplectic B-metric manifolds*.

Let $\{\xi; e_i\}$ $(i = 1, 2, \dots, 2n)$ be a basis of T_pM and let $\big(g^{ij}\big)$ be the inverse matrix of (g_{ij}). Then with F are associated the 1-forms θ, θ^*, ω, called *Lee forms*, defined by:

$$\theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i,\varphi e_j, z), \quad \omega(z) = F(\xi,\xi,z).$$

Now let us consider the case of the lowest dimension of the considered manifolds, i.e. $\dim M = 3$.

We introduce an almost contact B-metric structure (φ, ξ, η, g) on M defined by

$$\begin{gathered}\varphi e_1 = 0, \qquad \varphi e_2 = e_3, \qquad \varphi e_3 = -e_2, \qquad \xi = e_1,\\ \eta(e_1) = 1, \qquad \eta(e_2) = \eta(e_3) = 0,\end{gathered} \tag{1.2}$$

$$g(e_1,e_1) = g(e_2,e_2) = -g(e_3,e_3) = 1, \qquad g(e_i,e_j) = 0, \;\; i\neq j \in \{1,2,3\}. \tag{1.3}$$

The components of F, θ, θ^*, ω with respect to the *φ-basis* $\{e_1, e_2, e_3\}$ are denoted by $F_{ijk} = F(e_i, e_j, e_k)$, $\theta_k = \theta(e_k)$, $\theta^*_k = \theta^*(e_k)$, $\omega_k = \omega(e_k)$. According to [4], we have:

$$\begin{array}{lll}\theta_1 = F_{221} - F_{331}, & \theta_2 = F_{222} - F_{332}, & \theta_3 = F_{223} - F_{322},\\ \theta^*_1 = F_{231} + F_{321}, & \theta^*_2 = F_{223} + F_{322}, & \theta^*_3 = F_{222} + F_{332},\\ \omega_1 = 0, & \omega_2 = F_{112}, & \omega_3 = F_{113}.\end{array}$$

If F^s $(s = 1, 2, \ldots, 11)$ are the components of F in the corresponding basic classes $\mathcal{F}_s$ then: [4]

$$
\begin{aligned}
&F^1(x,y,z) = \left(x^2\theta_2 - x^3\theta_3\right)\left(y^2z^2 + y^3z^3\right),\\
&\qquad \theta_2 = F_{222} = F_{233}, \qquad \theta_3 = -F_{322} = -F_{333};\\
&F^2(x,y,z) = F^3(x,y,z) = 0;\\
&F^4(x,y,z) = \tfrac{1}{2}\theta_1\Big\{x^2\left(y^1z^2 + y^2z^1\right) - x^3\left(y^1z^3 + y^3z^1\right)\Big\},\\
&\qquad \tfrac{1}{2}\theta_1 = F_{212} = F_{221} = -F_{313} = -F_{331};\\
&F^5(x,y,z) = \tfrac{1}{2}\theta_1^*\big\{x^2\left(y^1z^3 + y^3z^1\right) + x^3\left(y^1z^2 + y^2z^1\right)\big\},\\
&\qquad \tfrac{1}{2}\theta_1^* = F_{213} = F_{231} = F_{312} = F_{321};\\
&F^6(x,y,z) = F^7(x,y,z) = 0;\\
&F^8(x,y,z) = \lambda\big\{x^2\left(y^1z^2 + y^2z^1\right) + x^3\left(y^1z^3 + y^3z^1\right)\big\},\\
&\qquad \lambda = F_{212} = F_{221} = F_{313} = F_{331};\\
&F^9(x,y,z) = \mu\big\{x^2\left(y^1z^3 + y^3z^1\right) - x^3\left(y^1z^2 + y^2z^1\right)\big\},\\
&\qquad \mu = F_{213} = F_{231} = -F_{312} = -F_{321};\\
&F^{10}(x,y,z) = \nu x^1\left(y^2z^2 + y^3z^3\right), \qquad \nu = F_{122} = F_{133};\\
&F^{11}(x,y,z) = x^1\big\{\left(y^2z^1 + y^1z^2\right)\omega_2 + \left(y^3z^1 + y^1z^3\right)\omega_3\big\},\\
&\qquad \omega_2 = F_{121} = F_{112}, \qquad \omega_3 = F_{131} = F_{113},
\end{aligned}
\tag{1.4}
$$

where $x = x^ie_i$, $y = y^je_j$, $z = z^ke_k$. Obviously, the class of 3-dimensional almost contact B-metric manifolds is

$$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}.$$

In [5] are considered three natural connections on an arbitrary $(M, \varphi, \xi, \eta, g)$, i.e. linear connections which preserve φ, ξ, η, g. They are called a φB-connection, a φ-canonical connection and a φKT-connection. The φB-connection is defined by

$$D_xy = \nabla_xy + \frac{1}{2}\big\{\left(\nabla_x\varphi\right)\varphi y + \left(\nabla_x\eta\right)y\cdot\xi\big\} - \eta(y)\nabla_x\xi. \tag{1.5}$$

The φ-canonical connection is determined by an identity for its torsion with respect to the structure tensors and the φKT-connection is characterized as the natural connection with totally antisymmetric torsion.

Since the considered manifold is 3-dimensional and the class $\mathcal{F}_3 \oplus \mathcal{F}_7$ is empty, then the φKT-connection does not exist and the φ-canonical connection coincides with the φB-connection.

In [5] is defined the square norm of $\nabla\varphi$ as follows

$$\|\nabla\varphi\|^2 = g^{ij}g^{ks}g\big(\left(\nabla_{e_i}\varphi\right)e_k, \left(\nabla_{e_j}\varphi\right)e_s\big). \tag{1.6}$$

An almost contact B-metric manifold having a zero square norm of $\nabla\varphi$ is called an *isotropic-cosymplectic B-metric manifold* ([5]). Obviously, the equality $\|\nabla\varphi\|^2 = 0$ is valid if $(M, \varphi, \xi, \eta, g)$ is a $\mathcal{F}_0$-manifold, but the inverse implication is not always true.

The Nijenhuis tensor N of the almost contact structure is defined as usual by $N = [\varphi, \varphi] + \mathrm{d}\eta \otimes \xi$, where $[\varphi, \varphi](x, y) = [\varphi x, \varphi y] + \varphi^2[x, y] - \varphi[\varphi x, y] - \varphi[x, \varphi y]$ for $[x, y] = \nabla_xy - \nabla_yx$ and $\mathrm{d}\eta$ is the exterior derivative of η. According to [6], the associated Nijenhuis tensor $\widehat{N}$ has the following form $\widehat{N} = \{\varphi, \varphi\} + (\mathcal{L}_\xi g) \otimes \xi$, where $\{\varphi, \varphi\}(x, y) = \{\varphi x, \varphi y\} + \varphi^2\{x, y\} - \varphi\{\varphi x, y\} - \varphi\{x, \varphi y\}$ for $\{x, y\} = \nabla_xy + \nabla_yx$ and $\mathcal{L}_\xi g$ is the Lie derivative of g with respect to ξ.

The corresponding tensors of type (0,3) on $(M, \varphi, \xi, \eta, g)$ are determined by $N(x, y, z) = g\left(N(x, y), z\right)$ and $\widehat{N}(x, y, z) = g(\widehat{N}(x, y), z)$. According to [6], it is

known that the tensors $N(x,y,z)$ and $\widehat{N}(x,y,z)$ are expressed by F as follows

$$
\begin{aligned}
N(x,y,z) &= F(\varphi x,y,z) - F(x,y,\varphi z) + \eta(z)F(x,\varphi y,\xi)\\
&\quad - F(\varphi y,x,z) + F(y,x,\varphi z) - \eta(z)F(y,\varphi x,\xi),\\
\widehat{N}(x,y,z) &= F(\varphi x,y,z) - F(x,y,\varphi z) + \eta(z)F(x,\varphi y,\xi)\\
&\quad + F(\varphi y,x,z) - F(y,x,\varphi z) + \eta(z)F(y,\varphi x,\xi).
\end{aligned}
\tag{1.7}
$$

Let $R = [\nabla,\nabla] - \nabla_{[\ ,\ ]}$ be the curvature (1,3)-tensor of ∇ and the corresponding curvature $(0,4)$-tensor be denoted by the same letter: $R(x,y,z,w) = g(R(x,y)z,w)$. The following properties are valid in general:

$$
\begin{aligned}
&R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z),\\
&R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0.
\end{aligned}
\tag{1.8}
$$

The Ricci tensor ρ and the scalar curvature τ for R and g as well as their associated quantities are defined as follows

$$
\begin{aligned}
&\rho(y,z) = g^{ij}R(e_i,y,z,e_j), \qquad \rho^*(y,z) = g^{ij}R(e_i,y,z,\varphi e_j),\\
&\tau = g^{ij}\rho(e_i,e_j), \quad \tau^* = g^{ij}\rho^*(e_i,e_j), \quad \tau^{**} = g^{ij}\rho^*(e_i,\varphi e_j).
\end{aligned}
\tag{1.9}
$$

Each non-degenerate 2-plane α in T_pM with respect to g and R has the following sectional curvature

$$
k(\alpha;p) = \frac{R(x,y,y,x)}{g(x,x)g(y,y)},
\tag{1.10}
$$

where $\{x,y\}$ is an orthogonal basis of α.

A 2-plane α is said to be a *φ-holomorphic section* (respectively, a *ξ-section*) if $\alpha = \varphi\alpha$ (respectively, $\xi \in \alpha$).

2. Space-like hyperspheres in $\mathbb{R}^{3,1}$

In this section we consider a hypersurface of the Lorentz-Minkowski space $\mathbb{R}^{3,1}$. Let $\langle\cdot,\cdot\rangle$ be the Lorentzian inner product, i.e.

$$
\langle x,y\rangle = x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4,
$$

where $x(x^1,x^2,x^3,x^4)$, $y(y^1,y^2,y^3,y^4)$ are arbitrary vectors in $\mathbb{R}^{3,1}$. Let us consider a space-like hypersphere S^3_1 at the origin with a real radius r identifying the point p in $\mathbb{R}^{3,1}$ with its position vector z, i.e.

$$
\langle z,z\rangle = r^2.
$$

It is parameterized by

$$
z(r\cos u^1\cos u^2, r\cos u^1\sin u^2, r\sin u^1\cosh u^3, r\sin u^1\sinh u^3),
$$

where u^1,u^2,u^3 are real parameters such as $u^1 \neq \frac{k\pi}{2} (k\in\mathbb{Z})$, $u^2 \in [0;2\pi]$. Then for the local basic vectors $\partial_i = \frac{\partial z}{\partial u^i}$ we have the following

$$
\begin{aligned}
&\langle\partial_1,\partial_1\rangle = r^2, \quad \langle\partial_2,\partial_2\rangle = r^2\cos^2 u^1, \quad \langle\partial_3,\partial_3\rangle = -r^2\sin^2 u^1,\\
&\langle\partial_i,\partial_j\rangle = 0,\ i\neq j.
\end{aligned}
$$

By substituting $e_i = \frac{1}{\sqrt{|\langle\partial_i,\partial_i\rangle|}}\partial_i$ we obtain a basis $\{e_i\}$, $i\in\{1,2,3\}$ as follows

$$
e_1 = \tfrac{1}{r}\partial_1, \qquad e_2 = \tfrac{\varepsilon_1}{r\cos u^1}\partial_2, \qquad e_3 = \tfrac{\varepsilon_2}{r\sin u^1}\partial_3,
\tag{2.1}
$$

where $\varepsilon_1 = \operatorname{sgn}(\cos u^1)$, $\varepsilon_2 = \operatorname{sgn}(\sin u^1)$. We equip it with an almost contact structure determined as in (1.2). The metric on the hypersurface, denoted by g, is

the restriction of $\langle\cdot,\cdot\rangle$ on the sphere. Then $\{e_1,e_2,e_3\}$ is an orthonormal φ-basis on the tangent space $T_pS^3_1$ at $p\in S^3_1$, i.e. for $g_{ij}=g(e_i,e_j)$, $i,j\in\{1,2,3\}$, we have (1.3). Thus, we get that $(S^3_1,\varphi,\xi,\eta,g)$ is a 3-dimensional almost contact B-metric manifold.

By virtue of (2.1) we obtain the commutators of the basic vectors e_i

$$[e_1,e_2]=\tfrac{1}{r}\tan u^1 e_2,\qquad [e_1,e_3]=-\tfrac{1}{r}\cot u^1 e_3,\qquad [e_2,e_3]=0. \tag{2.2}$$

Using the well-known Koszul identity for ∇ of g we get

$$\begin{array}{ll} \nabla_{e_2}e_1=-\tfrac{1}{r}\tan u^1 e_2, & \nabla_{e_2}e_2=\tfrac{1}{r}\tan u^1 e_1,\\ \nabla_{e_3}e_1=\tfrac{1}{r}\cot u^1 e_3, & \nabla_{e_3}e_3=\tfrac{1}{r}\cot u^1 e_1 \end{array} \tag{2.3}$$

and the other components are zero.

Let us compute the components of the natural connection denoted by D in (1.5). Then, using (1.2), (1.3), (1.5), (2.3), we establish that

$$D_{e_i}e_j=0,\qquad i,j\in\{1,2,3\}. \tag{2.4}$$

According to (1.2), (1.3) and (2.3), we obtain the value of the square norm of $\nabla\varphi$ as follows

$$\|\nabla\varphi\|^2=-\frac{2}{r^2}(\tan^2 u^1+\cot^2 u^1). \tag{2.5}$$

Taking into account (1.2), (1.3) and (2.3), we compute the components F_{ijk} of F with respect to the basis $\{e_1,e_2,e_3\}$. They are

$$F_{213}=F_{231}=-\frac{1}{r}\tan u^1,\qquad F_{312}=F_{321}=\frac{1}{r}\cot u^1 \tag{2.6}$$

and the other components of F are zero.

Using (1.7) and (2.6), we find the basic components $N_{ijk}=N(e_i,e_j,e_k)$ and $\widehat{N}_{ijk}=\widehat{N}(e_i,e_j,e_k)$ of the Nijenhuis tensor and its associated tensor, respectively,

$$\begin{array}{l} N_{122}=-N_{212}=N_{133}=-N_{313}=-\tfrac{1}{r}(\cot u^1+\tan u^1),\\ \widehat{N}_{122}=\widehat{N}_{212}=\widehat{N}_{133}=\widehat{N}_{313}=\tfrac{1}{r}(\cot u^1+\tan u^1),\\ \widehat{N}_{221}=-\widehat{N}_{331}=-\tfrac{4}{r}\tan u^1, \end{array}$$

as well as their square norms, according to (1.6), as follows

$$\begin{array}{l} \|N\|^2=\tfrac{4}{r^2}(\cot^2 u^1+\tan^2 u^1+2),\\ \|\widehat{N}\|^2=\tfrac{4}{r^2}(\cot^2 u^1+9\tan^2 u^1+2). \end{array} \tag{2.7}$$

Bearing in mind (1.4) and (2.6), we establish the equality

$$F(x,y,z)=(F^5+F^9)(x,y,z), \tag{2.8}$$

where F^5 and F^9 are the components of F in the basic classes $\mathcal{F}_5$ and $\mathcal{F}_9$, respectively. The nonzero components of F^5 and F^9 with respect to $\{e_1,e_2,e_3\}$ are the following

$$\begin{array}{l} F^5_{213}=F^5_{231}=F^5_{312}=F^5_{321}=\tfrac{1}{2}\theta^*_1=\tfrac{1}{2r}(\cot u^1-\tan u^1),\\ F^9_{213}=F^9_{231}=-F^9_{312}=-F^9_{321}=\mu=-\tfrac{1}{2r}(\cot u^1+\tan u^1). \end{array} \tag{2.9}$$

Let us remark that the above components of F^5 and F^9 are nonzero for all values of u^1 in its domain. By virtue of (2.8), (2.9) and (1.1), we get that

$$d\eta=0,\qquad \nabla_\xi\xi=0. \tag{2.10}$$

Using (1.3), (2.2) and (2.3), we compute the components $R_{ijk\ell} = R(e_i, e_j, e_k, e_\ell)$ of the curvature tensor R with respect to $\{e_1, e_2, e_3\}$. The nonzero components are given by the following ones and the symmetries of R in (1.8)

$$R_{1221} = -R_{1331} = -R_{2332} = \frac{1}{r^2}. \tag{2.11}$$

By virtue of (1.3), (1.9) and (2.11), the basic components $\rho_{jk} = \rho(e_j, e_k)$ and $\rho^*_{jk} = \rho^*(e_j, e_k)$ of the Ricci tensor ρ and its associated tensor ρ^*, respectively, as well as the values of the scalar curvature τ and its associated curvatures τ^*, τ^{**} are the following

$$\begin{array}{l} \rho_{11} = \rho_{22} = -\rho_{33} = \frac{2}{r^2}, \quad \rho^*_{23} = \rho^*_{32} = \frac{1}{r^2}, \\ \tau = \frac{6}{r^2}, \qquad \tau^* = 0, \qquad \tau^{**} = \frac{2}{r^2}. \end{array}$$

Moreover, using (1.3), (1.10) and (2.11), we obtain the basic sectional curvatures $k_{ij} = k(e_i, e_j)$ determined by the basis $\{e_i, e_j\}$ of the corresponding 2-plane as follows

$$k_{12} = k_{13} = k_{23} = \frac{1}{r^2}. \tag{2.12}$$

Let us remark that (1.3), (2.11) and (2.12) imply the following form of the curvature tensor

$$R(x, y, z, w) = \frac{1}{r^2}\{g(y, z)g(x, w) - g(x, z)g(y, w)\}. \tag{2.13}$$

Bearing in mind the above results, we establish the truthfulness of the following

Theorem 2.1. *Let $(S^3_1, \varphi, \xi, \eta, g)$ be the space-like sphere in the Lorentz-Minkowski space $\mathbb{R}^{3,1}$ equipped with an almost contact B-metric structure. Then*

(1) *the manifold is in the class $\mathcal{F}_5 \oplus \mathcal{F}_9$ but it belongs neither to $\mathcal{F}_5$ nor $\mathcal{F}_9$ and it is not an isotropic-cosymplectic B-metric manifold;*
(2) *the φB-connection which coincides with the φ-canonical connection vanishes in the basis $\{e_1, e_2, e_3\}$;*
(3) *the square norm of $\nabla\varphi$ is negative;*
(4) *the square norms of the Nijenhuis tensor and its associated are positive;*
(5) *the contact form η is closed and the integral curves of ξ are geodesic;*
(6) *the manifold is a space-form with positive constant sectional curvature.*

Proof. The proposition (1) follows from (2.5), (2.8) and (2.9). The truthfulness of the propositions (2), (3), (4), (5), (6) follows from (2.4), (2.5), (2.7), (2.10), (2.13), respectively. □

3. Time-like hyperspheres in $\mathbb{R}^{2,2}$

In [3], it is considered a unit time-like hypersphere S in $(\mathbb{R}^{2n+2}, J, G)$, where $\mathbb{R}^{2n+2}$ is a complex Riemannian manifold with a canonical complex structure J and a Norden metric G. There is introduced an almost contact B-metric structure on S in appropriate way by means of J and G. The constructed hypersphere with the considered structure belongs to the class $\mathcal{F}_4 \oplus \mathcal{F}_5$.

In this section we use a different approach for equipping a time-like hypersphere in $\mathbb{R}^{2n+2}$ for $n = 1$ with an almost contact B-metric structure.

Let us consider the neutral pseudo-Euclidean 4-space $\mathbb{R}^{2,2}$. Let $\langle\cdot,\cdot\rangle$ be the inner product defined by

$$\langle x, y\rangle = x^1y^1 + x^2y^2 - x^3y^3 - x^4y^4$$

for arbitrary vectors $x(x^1,x^2,x^3,x^4)$, $y(y^1,y^2,y^3,y^4)$ in $\mathbb{R}^{2,2}$. Let us consider a time-like hypersphere H_1^3 at the origin with a real radius r identifying the point p in $\mathbb{R}^{2,2}$ with its position vector z, i.e.

$$\langle z,z\rangle=-r^2.$$

It is parameterized by

$$z(r\sinh u^1\cos u^2, r\sinh u^1\sin u^2, r\cosh u^1\cos u^3, r\cosh u^1\sin u^3),$$

where $u^1,u^2,u^3\in\mathbb{R}$ such as $u^1\neq 0$. Then, for the local basic vectors ∂_i, we have the following

$$\begin{array}{l}\langle\partial_1,\partial_1\rangle=r^2,\quad \langle\partial_2,\partial_2\rangle=r^2\sinh^2 u^1,\quad \langle\partial_3,\partial_3\rangle=-r^2\cosh^2 u^1,\\ \langle\partial_i,\partial_j\rangle=0,\ i\neq j.\end{array}$$

Similarly as in the previous section, we substitute $e_i=\frac{1}{\sqrt{|\langle\partial_i,\partial_i\rangle|}}\partial_i$ and we obtain an orthonormal basis $\{e_i\}$, $i\in\{1,2,3\}$, as follows

$$e_1=\tfrac{1}{r}\partial_1,\qquad e_2=\tfrac{\varepsilon}{r\sinh u^1}\partial_2,\qquad e_3=\tfrac{1}{r\cosh u^1}\partial_3,$$

where $\varepsilon=\mathrm{sgn}(u^1)$. As for S_1^3, we introduce an almost contact B-metric structure on H_1^3 determined by (1.2) and (1.3). Hence, we get that $(H_1^3,\varphi,\xi,\eta,g)$ is a 3-dimensional almost contact B-metric manifold.

By similar way as for S_1^3 we obtain successively the following results:

$$[e_1,e_2]=-\tfrac{1}{r}\coth u^1 e_2,\qquad [e_1,e_3]=-\tfrac{1}{r}\tanh u^1 e_3,\qquad [e_2,e_3]=0,$$

$$\begin{array}{ll}\nabla_{e_2}e_1=\tfrac{1}{r}\coth u^1 e_2, & \nabla_{e_2}e_2=-\tfrac{1}{r}\coth u^1 e_1,\\ \nabla_{e_3}e_1=\tfrac{1}{r}\tanh u^1 e_3, & \nabla_{e_3}e_3=\tfrac{1}{r}\tanh u^1 e_1,\end{array}$$

$$D_{e_i}e_j=0,\qquad i,j\in\{1,2,3\}, \tag{3.1}$$

$$\|\nabla\varphi\|^2=-\tfrac{2}{r^2}(\tanh^2 u^1+\coth^2 u^1), \tag{3.2}$$

$$\begin{array}{c}F_{213}=F_{231}=\tfrac{1}{r}\coth u^1,\qquad F_{312}=F_{321}=\tfrac{1}{r}\tanh u^1,\\ N_{122}=-N_{212}=N_{133}=-N_{313}=\tfrac{2}{r\sinh 2u^1},\\ \widehat{N}_{122}=\widehat{N}_{212}=\widehat{N}_{133}=\widehat{N}_{313}=-\tfrac{2}{r\sinh 2u^1},\\ \widehat{N}_{221}=-\widehat{N}_{331}=\tfrac{2}{r}(\coth u^1+\tanh u^1),\end{array}$$

$$\begin{array}{l}\|N\|^2=\tfrac{4}{r^2}(\coth^2 u^1+\tanh^2 u^1+2),\\ \|\widehat{N}\|^2=\tfrac{4}{r^2}(3\coth^2 u^1+3\tanh^2 u^1+2),\end{array} \tag{3.3}$$

$$F(x,y,z)=(F^5+F^9)(x,y,z), \tag{3.4}$$

$$\begin{array}{l}F^5_{213}=F^5_{231}=F^5_{312}=F^5_{321}=\tfrac{1}{2}\theta_1^*=\tfrac{1}{2r}(\coth u^1+\tanh u^1),\\ F^9_{213}=F^9_{231}=-F^9_{312}=-F^9_{321}=\mu=\tfrac{1}{2r}(\coth u^1-\tanh u^1),\end{array} \tag{3.5}$$

$$d\eta=0,\qquad \nabla_\xi\xi=0, \tag{3.6}$$

$$R_{1221}=-R_{1331}=-R_{2332}=k_{12}=k_{13}=k_{23}=-\tfrac{1}{r^2}, \tag{3.7}$$

$$\begin{array}{l}\rho_{11}=\rho_{22}=-\rho_{33}=-\tfrac{2}{r^2},\qquad \rho^*_{23}=\rho^*_{32}=-\tfrac{1}{r^2},\\ \tau=-\tfrac{6}{r^2},\qquad \tau^*=0,\qquad \tau^{**}=-\tfrac{2}{r^2}.\end{array}$$

Similarly to the case of S_1^3, the obtained results could be interpreted in the following

Theorem 3.1. *Let $(H_1^3, \varphi, \xi, \eta, g)$ be the time-like sphere in the space $\mathbb{R}^{2,2}$ equipped with an almost contact B-metric structure. Then*

(1) *the manifold is in the class $\mathcal{F}_5 \oplus \mathcal{F}_9$ but it belongs neither to $\mathcal{F}_5$ nor $\mathcal{F}_9$ and it is not an isotropic-cosymplectic B-metric manifold;*
(2) *the φB-connection which coincides with the φ-canonical connection vanishes in the basis $\{e_1, e_2, e_3\}$;*
(3) *the square norm of $\nabla\varphi$ is negative;*
(4) *the square norms of the Nijenhuis tensor and its associated are positive;*
(5) *the contact form η is closed and the integral curves of ξ are geodesic;*
(6) *the manifold is a space-form with negative constant sectional curvature.*

Proof. The proposition (1) follows from (3.2), (3.4) and (3.5). The truthfulness of the propositions (2), (3), (4), (5), (6) follows from (3.1), (3.2), (3.3), (3.6), (3.7), respectively. □

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